Unit III ① LR TEST

$$f(x, \mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$\therefore L = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$= \left(\frac{1}{2\pi}\right)^{n/2} \cdot \left[\frac{1}{(\sigma^2)}\right]^{n/2} \cdot e^{-\frac{1}{2}\sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right)^2}$$

$$\log L = -\frac{n}{2}\log 2\pi - \frac{n}{2}\log\sigma^2 - \frac{1}{2}\sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right)^2$$

The maximum likelihood estimates of μ and σ^2 are as follows.

$$\frac{\partial \log L}{\partial \mu} = 0 \Rightarrow 0 - 0 + \frac{1}{\cancel{2}} \cancel{2} \sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma^2}\right) = 0$$

$$\Rightarrow \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i-\mu) = 0$$

$$\Rightarrow \sum_{i=1}^{n}(x_i-\mu) = 0$$

$$\Rightarrow \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \mu = 0$$

$$\Rightarrow n\bar{x} - n\mu = 0 \quad \left[\because \bar{x} = \frac{\sum x_i}{n}\right]$$

$$\Rightarrow \bar{x} = \hat{\mu}$$

$$\frac{\partial \log L}{\partial \sigma^2} = 0$$

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$$\Rightarrow -\frac{n}{2}\cdot\frac{1}{\sigma^2}+\sum_{i=1}^{n}(x_i-\mu)^2\cdot\frac{1}{2(\sigma^2)^2}=0$$

$\left\{ (\sigma^2)^{-1} \right.$ if we are differentiating

$$\left. (-1)(\sigma^2)^{-2} = -\frac{1}{(\sigma^2)^2} \right\}$$

$$\therefore \frac{\sum_{i=1}^{n}(x_i-\mu)^2}{\cancel{2}(\sigma^2)^{\cancel{2}}} = \frac{n}{\cancel{2}\sigma^{\cancel{2}}}$$

$$\frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2 = \hat{\sigma}^2 = s^2$$

In the parameter space ω L becomes

$$L = \left(\frac{1}{2\pi\sigma^2}\right)^{n/2}\cdot e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2}$$

by substituting $\hat{\mu}=\bar{x}$ & $\sigma^2 = s^2$

$$L = \left(\frac{1}{2\pi s^2}\right)^{n/2}\cdot e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\bar{x})^2}$$

$$= \left(\frac{1}{2\pi s^2}\right)^{n/2}\cdot e^{-\frac{1}{2\cancel{\sigma^2}}n\cdot\cancel{\sigma^2}}$$

$$= \left(\frac{1}{2\pi s^2}\right)^{n/2}\cdot e^{-n/2}$$

For $\mu = \mu_0$

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\mu_0)^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x}+\bar{x}-\mu_0)^2$$

$$= \frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2 + \frac{1}{n}\sum_{i=1}^{n}(\bar{x}-\mu_0)^2$$

$$= \frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2 + \frac{\cancel{n}(\bar{x}-\mu_0)^2}{\cancel{n}}$$

the cross product term vanishes

since $\Sigma(x_i-\bar{x})(\bar{x}-\mu_0) = (\bar{x}-\mu_0)\Sigma(x_i-\bar{x}) = 0$

$$\therefore \hat{\sigma}^2 = \sigma^2 + (\bar{x}-\mu_0)^2 = S_0^2 \text{ (say)}$$

$$L(\hat{\omega}_0) = \left(\frac{1}{2\pi S_0^2}\right)^{n/2} \cdot e^{-n/2}$$

The likelihood ratio $\lambda = \dfrac{L(\hat{\omega}_0)}{L(\hat{\omega})}$

$$= \frac{\left(\frac{1}{2\pi S_0^2}\right)^{n/2} \cdot \cancel{e^{-n/2}}}{\left(\frac{1}{2\pi s^2}\right)^{n/2} \cdot \cancel{e^{-n/2}}} = \frac{(\cancel{2\pi} s^2)^{n/2}}{(\cancel{2\pi} S_0^2)^{n/2}}$$

$$= \left[\frac{s^2}{s^2 + (\bar{x} - \mu_0)^2} \right]^{n/2}$$

$$= \left[\frac{1}{1 + \frac{(\bar{x} - \mu_0)^2}{s^2}} \right]^{n/2}$$

we know the statistic

$$t = \frac{\bar{x} - \mu_0}{S/\sqrt{n}} \sim t_{n-1}$$

where $S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2$

$$= \frac{n s^2}{n-1} \quad \left[\because s^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2 \right]$$

Thus $t = \frac{\bar{x} - \mu_0}{S/\sqrt{n}} = \frac{\bar{x} - \mu_0}{s/\sqrt{n-1}} \sim t_{n-1}$

$$\therefore \lambda = \frac{1}{\left(1 + \frac{t^2}{n-1}\right)^{n/2}} = \phi(t^2) \text{ (say)}$$

i.e $\frac{(\bar{x} - \mu_0)^2}{s^2/n-1} = t^2$

$$\frac{(\bar{x}-\mu_0)^2 (n-1)}{s^2} = t^2$$

$$\therefore \frac{(\bar{x}-\mu_0)^2}{s^2} = \frac{t^2}{n-1}$$

The likelihood ratio test for testing H_0 vs H_1 consists in finding a critical region of the type $0<\lambda<\lambda_0$.

in this case $\lambda = \phi(t)$ is a monotonic function of t^2. Now $t^2 = 0$ when $\lambda = 1$ and t^2 becomes infinite when $\lambda = 0$

i.e $$\left(1+\frac{t^2}{n-1}\right)^{-n/2} \leq \lambda_0$$

$$\Rightarrow \left(1+\frac{t^2}{n-1}\right)^{n/2} \geq (\lambda_0)^{-1}$$

$$\Rightarrow \frac{t^2}{n-1} \geq (\lambda_0)^{-2/n} - 1$$

$$\Rightarrow t^2 \geq (n-1)\left[\lambda_0^{-2/n} - 1\right] = A^2 \text{ (say)}$$

Thus the critical region may be

$$|t| = \left| \frac{\bar{x} - \mu_0}{S/\sqrt{n}} \right| \geq A$$

$$= \left| \frac{\sqrt{n}\,(\bar{x} - \mu_0)}{S} \right| \geq A$$

where the constant A is determined such that

$$P\left[\,|t| \geq A \,/\, H_0\right] = \alpha$$

Student's t distribution with $(n-1)$ d.o.f

if t distn with n d.o.f then

$$P[t > t_n(\alpha)] = \int_{t_n(\alpha)}^{\infty} f(t).dt = \alpha$$

Thus for testing $H_0: \mu = \mu_0$ vs $H_1: \mu \neq \mu_0$ we have two tailed t test as follows.

If $|t| = \left| \frac{\sqrt{n}\,(\bar{x} - \mu_0)}{S} \right| > t_{n-1}(\alpha/2)$ H_0 is rejected

& if $|t| < t_{n-1}(\alpha/2)$ H_0 may be accepted